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AN ALGEBRA FOR THE REALIZATION OF SWITCHING FUNCTION USING A CERTAIN TYPE OF MOS PACKAGE

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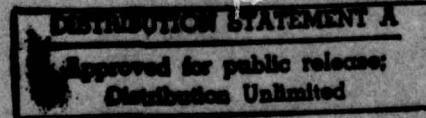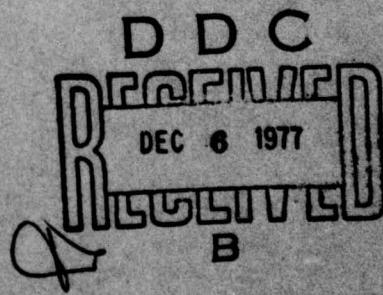
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Switching functions may be realized using a certain type of MOSFET circuit package, similar to the Fairchild 3102 package. This paper outlines the conditions that must be satisfied by a switching function so that it may be realized in a series-parallel form or in a bridge form using the least number of MOS elements in the package. Algorithms are given to obtain near optimal realizations when minimal realizations do not exist.

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by

Ravindra Nair and Gernot Metze

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Abstract

Switching functions may be realized using a certain type of
MOSFET circuit package, similar to the Fairchild 3102 package. This
paper outlines the conditions that must be satisfied by a switching
function so that it may be realized in a series-parallel form or in a
bridge form using the least number of MOS elements in the package.
Algorithms are given to obtain near optimal realizations when minimal
realizations do not exist.

1. INTRODUCTION

This paper studies the realizability of switching functions using a MOSFET circuit package of the form shown in Fig. 1. This type of package is commercially available, e.g., as the Fairchild 3102 integrated circuit where $n = 3$.
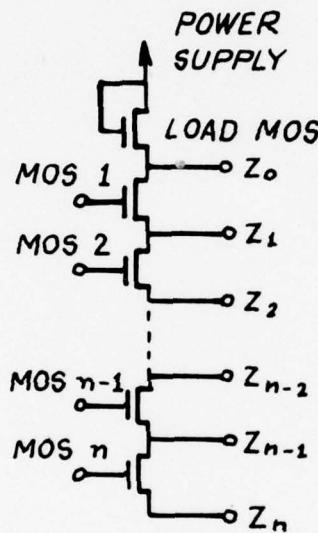
Fig. 1

The gates of the MOSFETs will be connected to the various input signals. The sources and drains of the MOSFETs may be interconnected in any manner with the restrictions that one of them must be connected to ground and that terminal z_0 must provide the output.

Interconnections of the pins $z_0, z_1, \ldots, z_n$ in various ways makes it possible to realize a host of switching functions. The limitations to realizing all switching functions of upto n variables arises from the fact that

a) the connection between gate i and gate $i+1$, for

$i = i, 2, \ldots, n-1$, is inseparable.

b) the output is restricted to be at pin z_0, i.e., at one

end of the chain formed by the MOSFETs.

Examples of 2 different functions realized by the package, with

$n = 3$, are shown in the following Figure:

$$G_1 = \overline{x + y + z}$$

$$G_2 = \overline{x(y+z)}$$
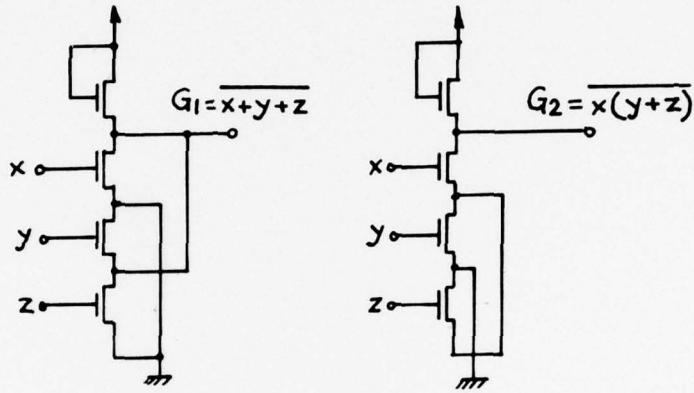
Fig. 2

The bilateral nature of the MOS gates makes them similar in their behaviour

to bilateral switches or relays. If each gate were replaced by a normally

open relay contact, then the only difference would be that the output of

the contact network would realize the complement of the output of the MOS

gate network. We will now proceed to determine the kinds of functions

that can be realized using this package, and algorithms for realizing

functions using the package.

2. DEFINITIONS

The basic postulates and theorems of switching algebra (e.g.

[KOH 70]) will be assumed. Letter variables a, b, c, ... will be used to

designate MOS gates (or contacts). Conjunction will be denoted by "·" or

simply by concatenation, disjunction by "+", and complementation by "-".

Letter variables and their complements are <u>literals</u>, and a conjunction of two or more literals forms a <u>term</u>. A term may also consist of a single literal. An <u>expression</u> may be a term, the conjunction of two expressions, or the disjunction of two expressions. A <u>function</u>, say $F = F(a,b)$, indicates that the variable F takes the value 0 or 1 depending on whether the expression $F(a,b)$ in variables a and b takes on the value 0 or the value 1.

A <u>minimal form</u> expression has the least number of literals among all equivalent expressions. A <u>least form</u> expression of n variables is a minimal form expression with exactly n literals, one for each variable.

A <u>unate function</u> has a minimal form expression with each variable expressed either in its true form only or in its complemented form only. In a <u>positive unate</u> function, all variables are expressed in their true forms only.

A network N is <u>series-parallel</u> if through each element (gate or relay) of N there is at least one path from the output terminal to the ground, not touching any junction twice, and no two of these paths pass through any element in opposite directions (this is a modification of the definition in [RIO 42]). A network which is not a series parallel network is a <u>bridge network</u>. A <u>bridge element</u> in a bridge network is an element belonging to a set of elements, the removal of which reduces the network to a series-parallel network, but the removal of any proper subset does not yield a series-parallel network (Modified from [MIL 58]).

In further discussion, bilateral relay switching networks will be considered. As seen already, the results may be applied equally effectively for MOS gate networks. Further, for convenience, the relay contact
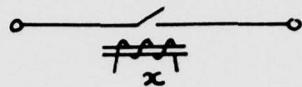
will be depicted as
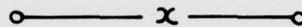
A function is <u>P-realizable</u> if it can be realized by a package of the form shown in Fig. 1. A function of k variables will be <u>minimally P-realizable</u> if it can be realized by a package of the form shown in Fig. 1, with no more than k elements in the chain, i.e., $n = k$. The function is assumed to be expressed in the least form.

The following observations can now be easily proved.

THEOREM 1:

$\forall n > 0$, $\exists k$, $0 < k \le n$, such that all functions of k or fewer variables are P-realizable with n elements in the package.

THEOREM 2:

$\forall k > 0$, $\exists n$, $n \ge k$, such that all functions of k or fewer variables are P-realizable with n elements in the package.

PROOF:

An algorithm will be provided to obtain such a realization.

Step 1:   Express the function in a sum-of-products form. Clearly, the total number of literals $\ge k$, say n (we will assume throughout that a k-variable function is not invariant in any variable). Associate one element with each literal.

Step 2:   For each term, connect all elements associated with literals
          in this term in series.

Step 3:   Connect all such series term chains in parallel, marking one
          end as the output and the other as ground. (Note that we are
          dealing with relay circuits. MOS circuits will be realized in
          a similar manner, taking output complementation into account.)

Step 4:   The required chain of n elements can now be easily identified,
          with the end of the chain grounded if the number of terms is
          odd, or connected back to the head of the chain otherwise.

                                                                    Q.E.D.

THEOREM 3:

       A minimally P-realizable function must be unate.   (Unateness is
only a necessary condition!)

COROLLARY 1:

       A minimally P-realizable function realized using only the true
form of the variables must be positive unate.

       In future discussions, only positive unate functions will be
considered. The extension to the entire class of unate functions is
immediate, if both true and complemented forms are available for variables.

3. SERIES-PARALLEL CIRCUITS

       A useful alternative definition for a series-parallel circuit
[MIL 58] is: A network is series-parallel if it is either a series or a
parallel connection of two series-parallel networks. A single element
(gate or relay) is also a series-parallel network.

We will now try to motivate the need for an efficient but systematic technique to determine the minimal P-realizability in the series parallel form of a positive unate function expressed in the least form.

Let us consider two sets P and C, where P consists of all the realizations possible using the package under consideration, while C consists of all the realizations possible in the conventional sense, without the two restrictions mentioned in section 1. Clearly, we can associate a mapping from the elements in P to those in C, in the sense that if $\alpha \in$ P maps into $\alpha' \in$ C then both α and α' have identical elements connected to every junction. In fact under the operations of series or parallel connection of elements in P and elements in C, we may define a homomorphism from P to C. Figure 3 exemplifies these concepts.

The P-realizations in Figs. 3(b) and 3(c) may be mapped to the conventional realization of Fig. 3(a). However the conventional realization in Fig. 3(d) which realizes the same function has no corresponding P-realization. Further the function realized in Fig. 3(e) has no corresponding P-realization even though there are many conventional realizations for the function. (In fact this function is the only such, for all positive unate functions of 6 or fewer variables, expressed in the least form.) Obviously, some method is needed to determine the minimal P-realizability of such functions without going through the painstaking task of listing all possible conventional realizations.

We will attempt to classify switching functions according to the form that they assume in a P-realization. For instance, the functions $F_1 = a$, $F_2 = ab$ and $F_3 = abc$ will be assumed to have similar forms, distinct

$$F = (a+b)c+d$$

(a)

$$F = (a+b)c+d$$

(b)

$$F = (a+b)c+d$$

(c)

$$F = (a+b)c+d$$

(d)

$$F = (a+b)c+(d+e)f$$

(e)
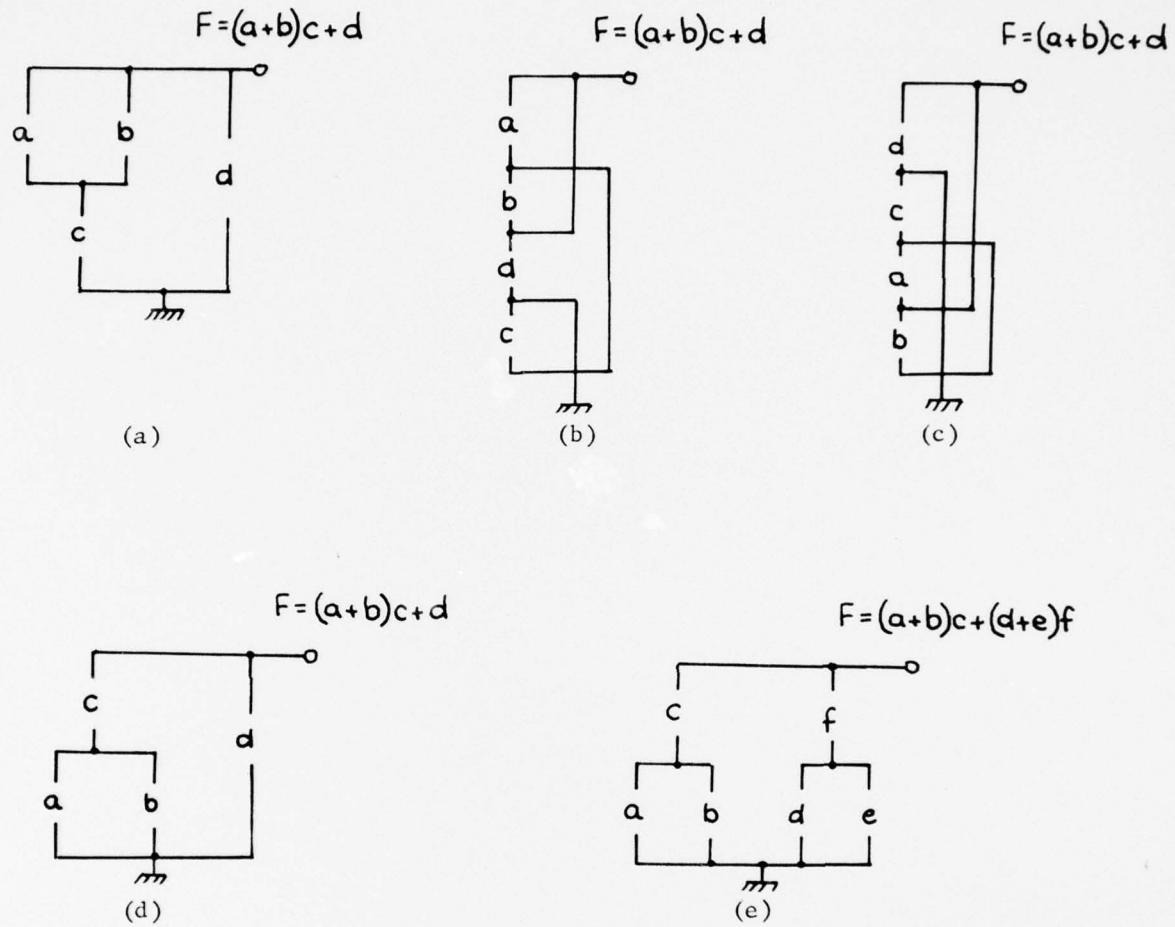
Fig. 3

from the form assumed by $F_4 = a+b$. This is because, for practical purposes, the behaviour of F_1, F_2 and F_3 in a more complex expression would be identical and distinguishable from the behaviour of F_4. For example, let us take the function $F = F_i + d$, where i may be 1, 2, 3 or 4. The final form would be as shown in Fig. 4(a) if i = 1, 2 or 3 and Fig. 4(b) if i = 4.
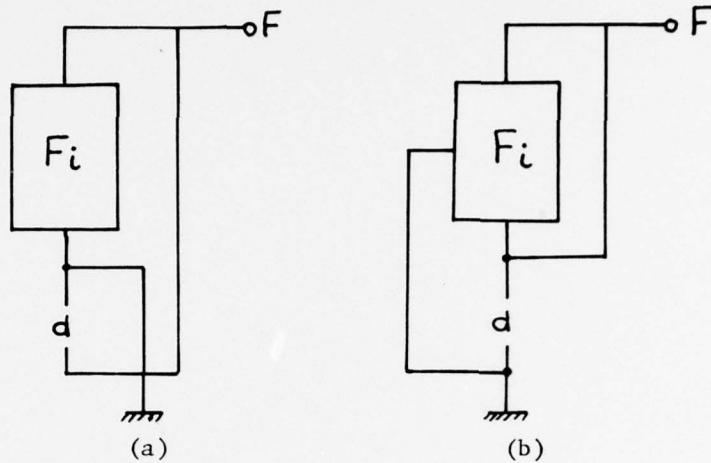
Fig. 4

Since every function in its least form is characterized by the absence of repeating literals, we can proceed as follows.

We replace every literal in the switching expression by the letter P. Thus $F_1 = P$, $F_2 = PP$, $F_3 = PPP$ and $F_4 = P + P$. Further, P, PP and PPP represent an equivalent functional form distinct from $P + P = S$, say. Clearly, it would be convenient to replace PP and PPP by P, an equivalent functional form, without loss of structural equivalence. What we have done then is simply to define a rule, which says, PP = P. By a similar argument, we notice that $P + P + P = P + S$ has the same functional form as P, while $P + P + P + P = S + S$ has the same functional form as S.

We are now in a position to define a set of rules - let us call it the P-S algebra - so that a unique minimal expression involving only P's

and S's may be obtained for a given switching expression in the least form. The final expression obtained will characterize the structure of the realization for the function.

Let F_1, F_2, F_3, etc. denote any expressions involving P and S. The laws of commutativity and associativity carry over directly from Boolean algebra to the P-S algebra.

1. Commutativity: $F_1 + F_2 = F_2 + F_1$

                         $F_1 F_2 = F_2 F_1$

2. Associativity: $F_1 + F_2 + F_3 = (F_1 + F_2) + F_3 = F_1 + (F_2 + F_3)$

3. S-idempotency: $SS = S$

                         $S + S = S$

4. Partial P-idempotency: $PP = P$

5. Absorption: $P + S = P$

6. Transformation: $P + P = S$ (This is more correctly a definition, not a rule)

An example to illustrate the application of the above rules is shown below: Let $F = (a(b+c)+d)(e+f(g+h)+k)+\ell\,(m+n)$. If G denotes the equivalent functional form then

$$G = (P(P+P)+P)(P+P(P+P)+P) + P(P+P)$$

$$= (PS+P)(P+PS+P) + PS \qquad \text{by Rule 6}$$

$$= (PS+P)(P+P+PS) + PS \qquad \text{by Rule 1}$$

$$= (PS+P)((P+P)+PS) + PS \qquad \text{by Rule 2}$$

$$= (PS+P)(S+PS) + PS \qquad \text{by Rule 6}$$

No further simplification is possible.

We can now associate forms for P and S as shown in Fig. 5, and observe that a P-realization is possible as long as there exists some numbering of the forms 1,2,3,...,k such that the terminal y of form i is

connected to the terminal x of $i+1$, $1 \le i < k$.
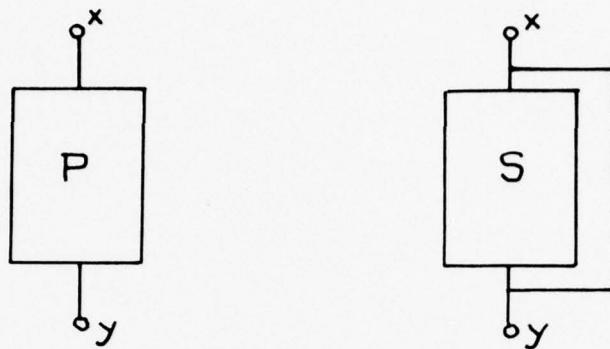
Fig. 5

For example, the functional form $(PS + P)P$ is realized as:
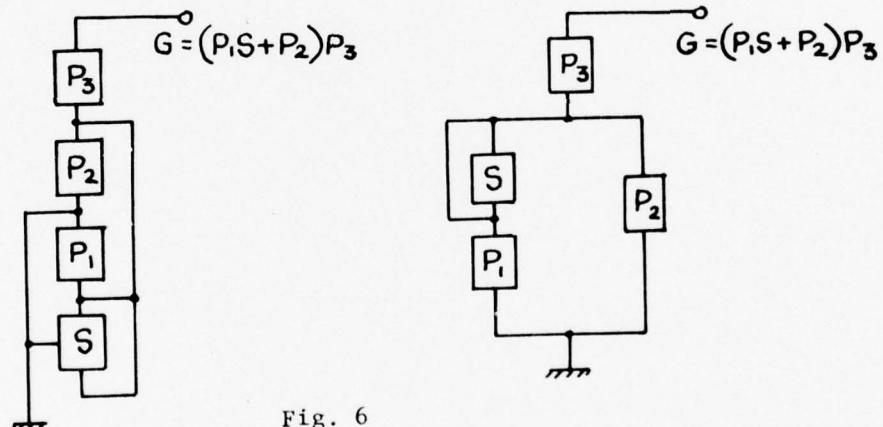
Fig. 6

We now define the <u>modified functional form</u> for a switching expression. If the minimized functional form for an expression is SG or PG, then the modified functional form is G. It is important to note that the suggested modifications can be carried out only on the entire expression for the function to be realized. It cannot be used to reduce subexpressions within expressions.

THEOREM 4: If the modified functional form for a switching expression is P-realizable, then the original functional form, without modifications, is also P-realizable.

PROOF: A general P-realizable form is shown in Fig. 7.
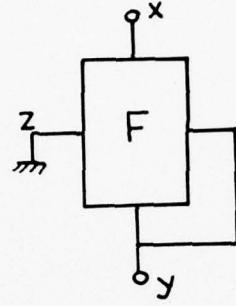
Fig. 7

Figure 8(a) and (b) show how the original functional form may be realized.
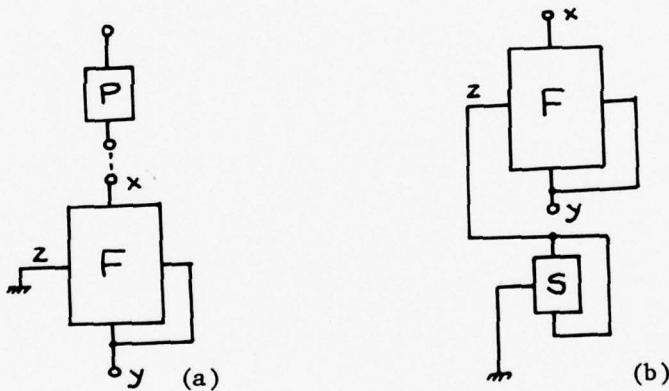

Fig. 8

Q.E.D.

Figure 8(b) shows the special case where the x-y chain connection mentioned earlier is violated. An example is in order to clarify this point.

H = (PS + P)S = GS is the functional form for the switching expression F = (a(b + c) + d)(e + f). G is realized as shown in Fig. 9(a), GS is realized as in 9(b) and the chain is clarified in Fig. 9(c).

(a)

(b)

(c)
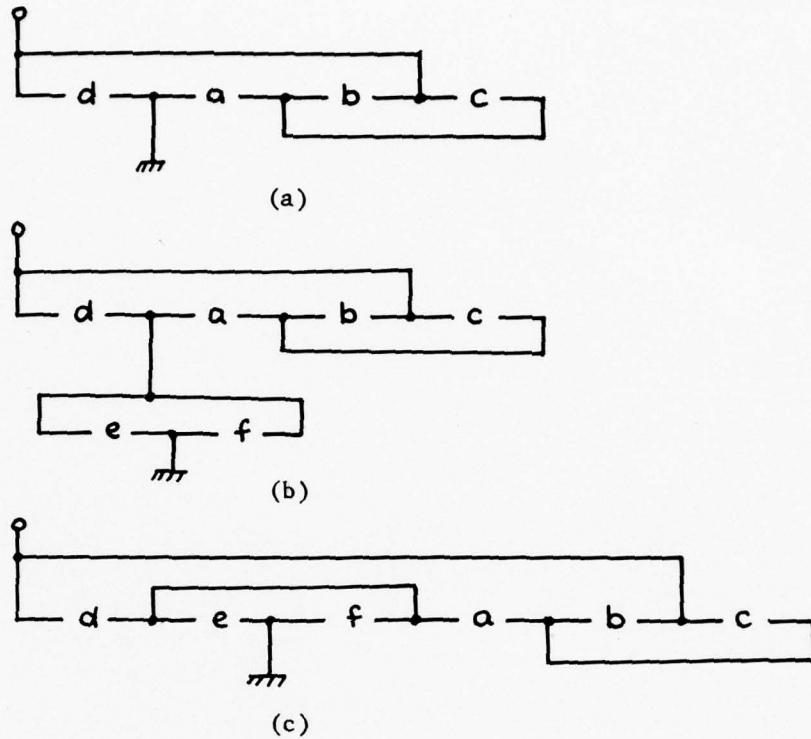
Fig. 9

THEOREM 5: A positive unate switching function expressed in the least form is minimally P-realizable if and only if there is no more than one occurrence of PS in the modified functional form for the corresponding expression.

PROOF: Sufficiency: The modified functional forms with at most one PS are P, S, PS + P and PS + S, each of which can be shown to be P-realizable quite easily. By Theorem 4 then the original functional forms should also be P-realizable, if the modified functional forms have at most one PS.

Necessity: It suffices to show that the functional forms of the type PS + PS + G and (PS + H)(PS + H) are not P-realizable, where G may be P,S or absent, and H may be P or S. This can again quite easily be shown, by

trial and error or otherwise. All other functional forms with 2 or more occurrences of PS "contain" one or both of these forms and hence are not P-realizable.

<div align="right">Q.E.D.</div>

The example shown below indicates a systematic way in which one would attempt to realize a series parallel circuit.

Let $\quad F = (a+b+c)(d+e+f+g) + (k\ell + (m+n)(o+p)) + r$

$$\therefore \quad G = (P_1+P_2+P_3)(P_4+P_5+P_6+P_7) + (P_8 P_9 + (P_{10}+P_{11})(P_{12}+P_{13})) + P_{14} \quad (1)$$

$$= (P_1+S_1)(S_2+S_3) + (P_{15}+S_4 S_5) + P_{14} \quad (2)$$

$$= (P_{16})(S_6) + (P_{15}+S_7) + P_{14} \quad (3)$$

$$= P_{16} S_6 + P_{17} + P_{14} \quad (4)$$

$$= P_{16} S_6 + S_8 , \quad \text{which is indeed P-realizable.} \quad (5)$$

Figure 10(a) shows the P-realization for the relation (5). From this realization, the realization of Fig. 10(b) is obtained for relation (3). We may proceed in this manner backwards to obtain the final realization of the function as shown in Fig. 10(c).

Theorem 5 leads to many interesting results, some of which are given below:

COROLLARY 2: Every positive unate function in the least form with 5 or fewer variables has a minimal P-realization in the series-parallel form.

COROLLARY 3: The only least form positive unate function of 6 variables that cannot be realized in a minimal P form is $a(b+c)+d(e+f)$.

Proof: The smallest functional form that is not P-realizable is $PS+PS$.
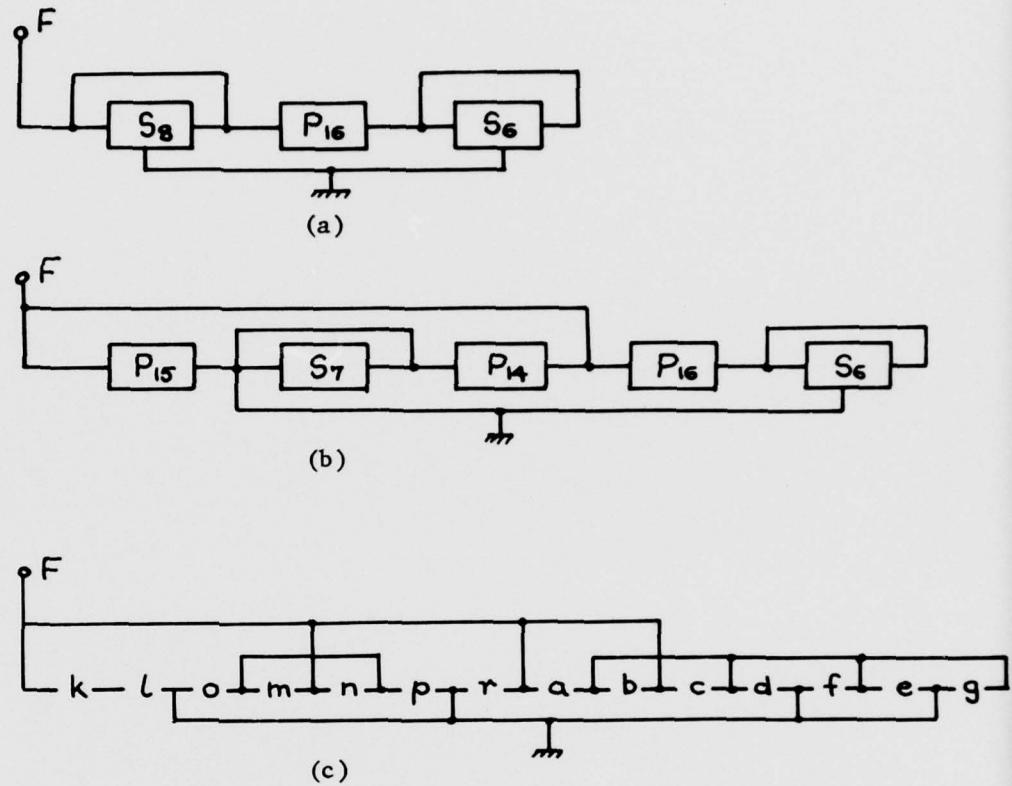
Fig. 10

COROLLARY 4: The least form positive unate functions of 7 variables that are not minimally P-realizable are:

$$ab(c+d)+e(f+g)$$

$$a(b(c+d)+e(f+g))$$

$$a(b+c)+d(e+f)+g, \text{ and}$$

$$a(bc+d)+e(f+g)$$

COROLLARY 5: For a given package, the last, i.e., nth, gate in the chain is grounded at its free end if and only if the minimized functional form for the function it realizes is P.

COROLLARY 6: For a given package, the last gate in the chain is connected to the output at its free end, if and only if the minimized functional form for the function it realizes is S.

Finally, it must be said that for switching expressions which are not in the least form, similar techniques may be used to determine a P-realization in the series parallel form. For instance, every duplicate occurrence of a variable may be considered a separate new variable. However, there is often the possibility of obtaining a minimal P-realization by using bridges. Such realizations will be studied in the following section.

4. GENERAL CIRCUITS WITH BRIDGES

A function like $F = ab + (c+f)d + ade + b(c+f)e$ is indeed in its minimal form (the number of literals cannot be reduced) but not in its least form. Nevertheless, there exists a minimal P-realization for the network. This realization is shown in Fig. 11 and the corresponding conventional form is shown in Fig. 12.
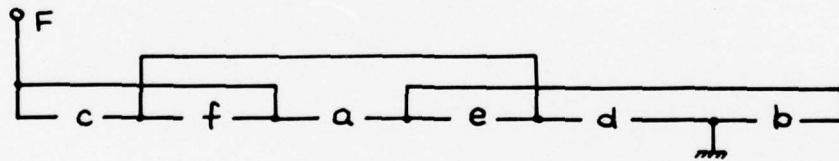

Fig. 11
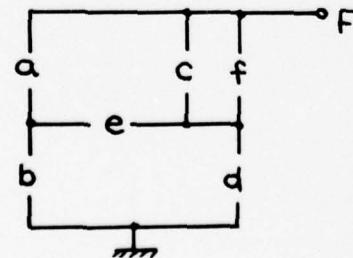
Fig. 12

A careful study indicates that the P-S algebra defined earlier is not of much use in bridged realizations. Extensive modification may yield a more useful, though probably inconvenient, vehicle for the algebraic analysis of bridged circuits. A more useful procedure involves concepts in graph theory.

Switching networks may be depicted as weighted linear graphs. The weights are Boolean functions attached to the elements of the graph. Following [MIL 58], an element consists of a line segment and its endpoints. A vertex is an endpoint of an element, and a linear graph is a collection of elements so that no two elements have a point in common which is not a vertex. A subgraph is a graph which contains a subset of the elements of the graph. A vertex and an element are incident with each other if the vertex is an endpoint of the element. The transmission function represents the transmission between two distinguished vertices in the graph, namely the output vertex and the ground vertex. All other vertices of the graph will be termed internal vertices.

The switching network may also be represented by a connection matrix [ARA 49, HOH 55]. For a graph with p vertices, the connection matrix is a p x p matrix. The matrix entries are Boolean functions. An entry c_{ij} represents the connection from vertex i to vertex j. The c_{ii} entries are all defined to be Boolean 1. If $c_{ij} = 0$, no direct connection exists between vertices i and j. If $c_{ij} = 0,1$, a literal, or a disjunction of literals for every c_{ij} of connection matrix, then the matrix is called a primitive connection matrix.

As shown in [MIL 57], the transmission function can be obtained as the output determinant of the connection matrix.

An <u>Euler path</u> (<u>Euler circuit</u>) is a path (circuit) that traverses each edge in a graph exactly once. The degree of a vertex in a graph is the number of edges incident with it.

The following results, Theorems 6 and 7, are famous results, the proofs of which can be obtained from any standard book dealing with graph theory [e.g. LIU 68].

<u>THEOREM 6</u>: The sum of the degrees of the vertices in a graph is an even number (self loops are ignored).

<u>COROLLARY 7</u>: In any graph, there is an even number of vertices of odd degree.

<u>THEOREM 7</u>: An undirected graph possesses an Euler path if and only if it is connected and has no, or exactly two, vertices of odd degree.

We are now in a position to state the necessary and sufficient conditions for determining the P-realizability of a given function.

<u>THEOREM 8</u>: A given positive unate function is minimally P-realizable if and only if

(a) it possesses at least one minimal conventional realization, and

(b) for some valid, minimal conventional realization, the number of internal vertices of odd degree is no more than one.

<u>PROOF</u>: <u>Sufficiency</u>: It is clear that if there is an Euler path in the conventional realization starting either at the output vertex or the ground vertex, then a P-realization is possible for the conventional realization. In the case where the Euler path starts from the ground vertex, the output and ground must be interchanged. In a graph which has an Euler path, and

which has nodes of odd degree, only the source and the destination of the Euler path have odd degrees. In our case the destination could be an internal vertex and hence condition (b) is sufficient for a P-realization. Necessity: Every P-realization is associated with a conventional realization. Hence the necessity of condition (a). If there are more than two internal vertices of odd degree, then an Euler path cannot exist by Theorem 7. If there are two internal vertices of odd degree, and both external vertices are of odd degree, again an Euler path cannot exist. If there are two-internal vertices of odd degree and none of the external vertices are of odd degree, an Euler path exists but cannot start at a terminal vertex.

This concludes the proof of the necessity and sufficiency of the conditions.   *Q.E.D.*

COROLLARY 8

A given positive unate function is minimally P-realizable if and only if in the primitive connection matrix for some realization of the function

> (a) a literal appears exactly twice, and

> (b) at most one row from the rows corresponding to the internal vertices, has an odd number of literals.

Note: (i) Condition (a) implies the existence of a minimal realization.

(ii) Because of symmetry of the matrix, a limit for the rows is also a limit for the columns.

An algorithm to determine a minimal P-realization for a given function will consist of the following steps:

Step 1:   Reduce the function in order to minimize the number of literals.

Step 2: If the number of literals equals the number of variables with a literal corresponding to every variable, then attempt a series-parallel minimal P-realization.

Step 3: If the number of literals is greater than the number of variables in the expression, then obtain a primitive connection matrix for the function.

Step 4: If the primitive connection matrix satisfies Corollary 8, then a minimal realization is obtained, else obtain another network or primitive connection matrix and repeat Step 4.

Clearly, Step 4 is a stumbling block because of the large number of matrices that can be associated with a given function. However, in most cases, the number of variables is not too high, so that once a minimal conventional realization is obtained, the realization satisfying Theorem 8 may be obtained by a visual scan.

An illustrative example follows:

Let $\quad F = af\ell + ag\ell + aeh + aek + bef\ell + beg\ell + bh + bk + cef\ell + ceg\ell$
$\qquad + ch + ck + def\ell + deg\ell + dh + dk.$

Using Miller's technique [MIL 58] or otherwise, we obtain

$\quad F = e(aP + QR) + aR + PQ,$ where

$\quad P = h + k, \quad Q = b + c + d, \quad R = (f + g)\ell,$ which satisfies Miller's bridge condition and has a connection matrix as shown:

$$
F = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\left[\begin{array}{cccc}
1 & a & Q & 0 \\
a & 1 & e & R \\
Q & e & 1 & P \\
0 & R & P & 1
\end{array}\right]
\end{array}
\quad \text{with terminal vertices 1 and 4}
$$

The primitive connection matrix for R is:

$$R = \begin{array}{c} \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{ccc} 5 & 6 & 7 \\ \left[\begin{array}{ccc} 1 & \ell & 0 \\ \ell & 1 & f+g \\ 0 & f+g & 1 \end{array}\right] \end{array}$$

A possible primitive connection matrix for F is:

$$F = \begin{array}{c} \\ 1 \\ 2 \\ 2' \\ 3 \\ 4 \end{array} \begin{array}{ccccc} 1 & 2 & 2' & 3 & 4 \\ \left[\begin{array}{ccccc} 1 & a & 0 & b+c+d & 0 \\ a & 1 & \ell & e & 0 \\ 0 & \ell & 1 & 0 & f+g \\ b+c+d & e & 0 & 1 & h+k \\ 0 & 0 & f+g & h+k & 1 \end{array}\right] \end{array}$$

with 1 and 4 as terminal vertices.

This matrix now *satisfies both conditions* of Corollary 8, there being only one internal vertex, vertex 2', which has an odd number of literals. Figures 13 and 14 show the minimal conventional realization and minimal P realization respectively.
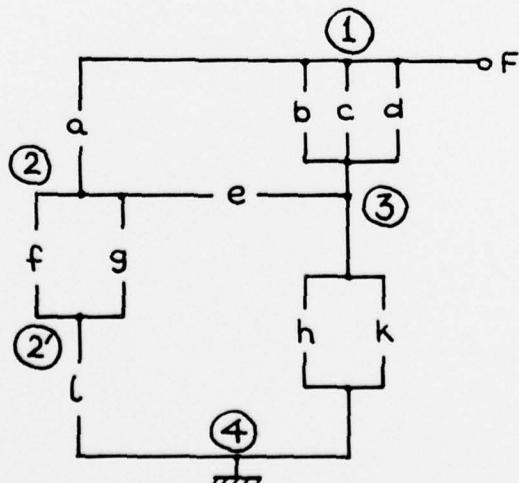

Fig. 13

Fig. 14

5. CONVERSION TO REALIZABLE CIRCUITS

Let us consider a function $F = abc(d+e) + f(g+h)$ which has a functional form $G = PS + PS$ and hence is not minimally P-realizable. With the knowledge that $(P+P)S + PS = SS + PS = S + PS$ is a realizable form, we could duplicate 3 elements to transform the function to $F = (abc+abc)(d+e) + f(g+h)$. However $P(S+P) + PS = PP + PS = P + PS$ is also a valid P-realizable form which requires the duplication of only one element, instead of three, using the transformation $F = abc(d+e+e) + f(g+h)$.

The need is felt, hence, for a technique which yields the best P-realization, given any function. Unfortunately, such a technique is not known at this point, but the algorithm to be presented in this section does provide optimal results in most cases and near optimal results in the others.

For the series-parallel case, the P-S algebra, as detailed in Section 3 is utilized, but each P and S is now subscripted by an integer which indicates the least number of elements that need to be duplicated to convert that S to a P or P to an S. When absorbing P's and S's, the subscripts will change according to the following rules:

1. $P_a P_b = P_{a+b}$

2. $S_a S_b = S_{a+b}$

3. $S_a + S_b = S_{min(a,b)}$

4. $P_a + S_b = P_{min(a,b)}$

5. $P_a + P_b = S_{min(a,b)}$

The initial subscript for all P's will be 1. An example follows:

$$F = abc(d + e) + f(g + h)$$

$$\Rightarrow \quad G = P_1 P_1 P_1 (P_1 + P_1) + P_1 (P_1 + P_1)$$

$$= P_2 P_1 (S_1) + P_1 (S_1)$$

$$= P_3 S_1 + P_1 S_1$$

It is thus seen instantly that changing S_1 to P_1 or P_1 to S_1 is more profitable than changing P_3 to S_3. The duplication strategy is hence suggested by this extension to the functional form of the expression.

Algorithm to obtain an optimal or near-optimal P-realization for a positive unate function in the least form:

Step 1: In the switching expression for the function, replace every variable by P_1, retaining the "+"s and "."s.

Step 2: Reduce the number of P's and S's in the resulting expression using commutativity, associativity and the preceding laws.

Step 3: Repeat step 2 until a stage is reached where none of the laws reduce the expression any further.

Step 4: If the functional form obtained satisfies Theorem 5, a P-realization is obtainable. Terminate the algorithm. Else step 5.

Step 5: Let $a_k(b_k)$ be the minimum of $(a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n)$, the subscripts of the n   PS's in the expression, ordered arbitrarily as $P_{a_i} S_{b_i}$, $1 \leq i \leq n$. Change P_{a_k} to S_{a_k} (S_{b_k} to P_{b_k}) by duplicating the appropriate $a_k(b_k)$ elements involved in the term $P_{a_k}(S_{b_k})$. Return to step 2 for further reduction.

Example:

$F = (a(b+c)(d+e) + gh(i+j) + k\ell m(n+o))(p+q) + n(s+t)(u+v)$

$G = (P_1(P_1+P_1)(P_1+P_1) + P_1 P_1(P_1+P_1) + P_1 P_1 P_1(P_1+P_1))(P_1+P_1) + P_1(P_1+P_1)(P_1+P_1)$

$\quad = (P_1 S_1 S_1 + P_2 S_1 + P_3 S_1)S_1 + P_1 S_1 S_1$

$\quad = (P_1 S_2 + P_2 S_1 + P_3 S_1)S_1 + P_1 S_2$

$\quad = (S_1 S_2 + P_2 S_1 + P_3 S_1)S_1 + P_1 S_2$, duplicating 'a'

$\quad = (S_3 + P_2 S_1 + P_3 S_1)S_1 + P_1 S_2$

$\quad = (S_3 + P_2 P_1 + P_3 S_1)S_1 + P_1 S_2$, duplicating 'j'

$\quad = (S_3 + P_3 + P_3 S_1)S_1 + P_1 S_2$

$\quad = (P_3 + P_3 S_1)S_1 + P_1 S_2$.

There are 2 options available.

(i) $G = (P_3 + P_3 P_1)S_1 + P_1 S_2$, duplicating 'o'

$\quad\quad = (P_3 + P_4)S_1 + P_1 S_2$

$\quad\quad = S_3 S_1 + P_1 S_2$

$\quad\quad = S_4 + P_1 S_2$, or

(ii) $G = (P_3 + P_3 S_1)S_1 + S_1 S_2$, duplicating 'r'

$\quad\quad = (P_3 + P_3 S_1)S_1 + S_3$.

Both yield valid functional forms for a P-realization. Figure 15 shows the conventional realization for approach (ii). The vertex marked 'x' is the only internal vertex having an odd degree.
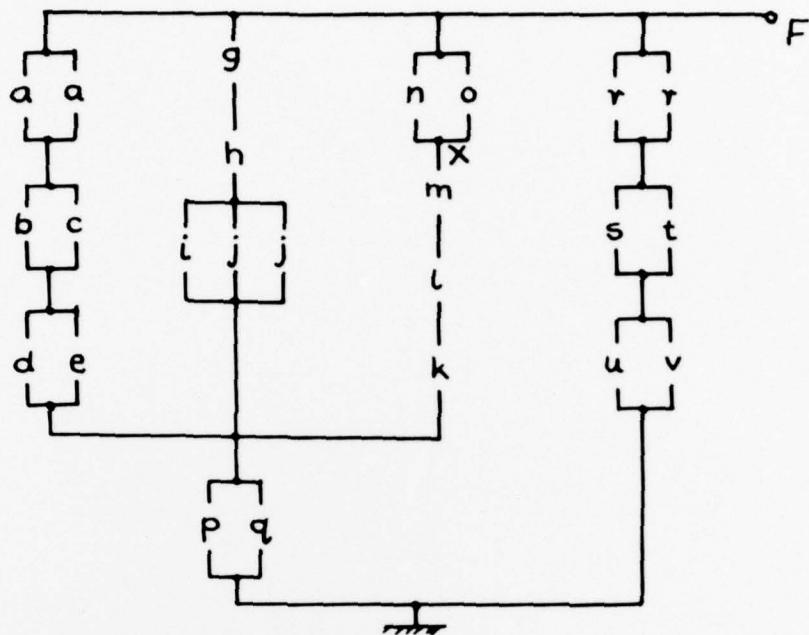
Fig. 15

The chain for the P-realization would be:

r-s-u-v-t-r-a-b-d-e-c-a-g-h-i-j-j-p-q-k-ℓ-m-n-o.

Given a specific functional form, an optimal realization can be found by exhausting the various reduction possibilities. However, when the final functional form contains 3 or more PS's the number of cases becomes too large to merit the search for optimality. In any case, the algorithm just described does lead to the optimal solution in most cases.

The situation is even more complex when the least form is not available. We now have to deal with a bridged network. A few heuristics that may be employed on the conventional realization, to make it P-realizable, will now be given.

Heuristic 1:

Determine the length of the shortest path [the length of a path is the number of elements in the path] from an internal vertex of odd degree to the terminal vertices. Duplicate the elements in this path. Repeat for other internal vertices of odd degree.

Heuristic 2:

Perform heuristic 1 and then undo the duplications in the path which had the largest number of duplications.

[One internal vertex is allowed to have an odd degree by the conditions of Theorem 8.]

Heuristic 3:

If while performing heuristic 2, the shortest path from an internal vertex a, passes through another internal vertex b, also of odd degree, then do not duplicate the elements from b to the terminal vertex.

[Both a and b thus are converted to nodes of even degree at the same time.]

Heuristic 4:

Let $V = \{v_1, v_2, \ldots, v_n\}$ be the set of internal vertices of odd degree. Let v_o and v_g be the terminal vertices. Among the paths from v_i to v_o, v_i to v_g and v_i to v_j for some $v_i \in V$, and every $v_j \in V$, $v_j \neq v_i$, find the path with the shortest length. Duplicate the elements in that path, and eliminate v_i from V. If the terminating vertex of the path also belongs to V, then eliminate that vertex from V also. Proceed in this manner with the reduced set V until V is empty. Determine the longest path duplicated terminating at one of the terminal vertices, and undo the duplications in that path.

Other heuristics may be constructed similarly. As the heuristic becomes more sophisticated, the P-realization gets more optimal accompanied by an increased cost of implementing the algorithm. A visual scan of the network could often help in achieving this optimality. In most practical situations, however, the mentioned heuristics would lead to an optimal realization.

6. DISCUSSION

The central idea of this paper is to indicate systematic techniques to realize functions using a specific organization described in Section 1. While it is difficult to determine exactly what fraction of all possible functions can be realized using the package minimally, it is clear from the results of sections 3 and 4 that for a function with a small number of variables, a large majority of the functions can indeed be realized using the package.

Extensions to the study and combinatorial analysis may be simplified by observing that the minimal series-parallel P-realizations that are possible are actually the strings in the language generated by the following grammar:

$$Z: = P|S|PS|PS + S|PS + P|P(PS + S)|P(PS + P)|S(PS + S)|S(PS + P)$$

$$S: = P + P|S + S|SS$$

$$P: = P + S|PP|a$$

We notice then that the technique used to check the P-realizability of a function in its least form is essentially a syntax-directed translation schema [AHO 72] using a bottom-up parser. Extension of this technique to the general bridge circuits does not seem possible.

Another interesting observation that can be made is that, by simply disconnecting the load MOSFET from the rest of the chain and providing an extra terminal in the package as shown in Fig. 16, the power of the package may be increased considerably. The reason for the extra power for the package arises from the fact that the output may now be taken from any of the gates in the chain instead of from one end of the chain only.
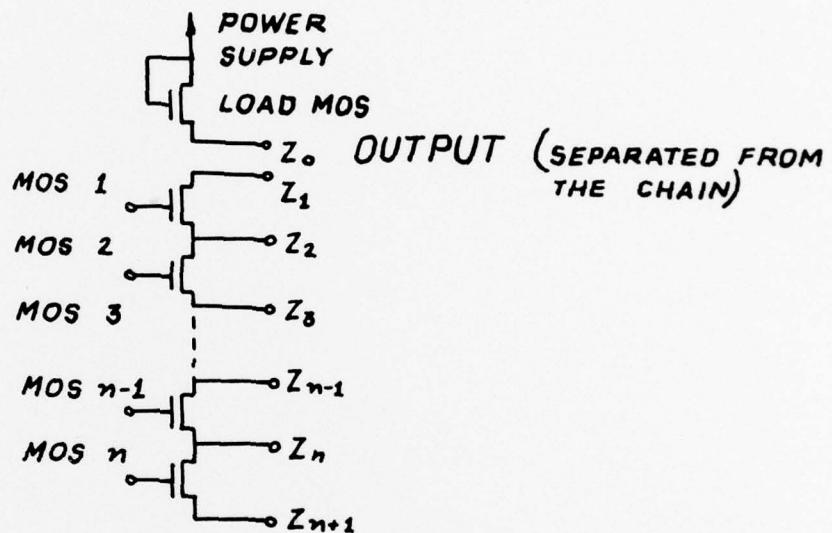

Fig. 16

With this restriction eliminated, the existence of an Euler path between any two vertices will ensure the existence of a realization in the new package (say, Q-realization). Theorem 8 may now be modified to:

THEOREM 9: A given positive unate function is minimally Q-realizable if and only if

(a) it possesses at least one minimal conventional realization, and

(b) for some valid, minimal conventional realization, the number of vertices of odd degree is no more than two.

It was mentioned earlier that when the number of variables becomes large, the procedures for giving a minimal P-realization or for obtaining an optimal transformation to a P-realizable form become quite complex. In any case, implementation difficulties will generally forbid the chaining of a large number of identical MOSFET gates, so that there is a limit placed on the number of variables that may exist in a function to be realized. The practical limitations arise from:

(a) variation of voltage levels at the output, due to variation of resistance in different network inter-connections, and

(b) limitation on the number of pins in an integrated circuit package.

Thus, it may be concluded that the systematic techniques developed in the earlier sections are very useful for determining P-realizability in almost every practical situation.

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